REASONING BY ANALOGY AND BY DIFFERENCE

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Abstract: The article presents a novel and extended analysis of reasoning by analogy. It delves deeper into the concept of 'domain,' derived from Wittgenstein's idea of categories, which serves as a fundamental aspect in defining relative analogies. Building upon this foundation, it closely examines what the literature refers to as 'determination rules,' and specifies their probabilistic and non-monotonic forms. A detailed exploration of the range of specific cases that can be encountered, introducing new concepts such as separation rules, counter-determination rules, and counter-separation rules, is proposed. Subsequently, we illustrate how this set of rules enables a unified set of inference schemes of analogical reasoning. This leads to address examples typically treated as independent and specific instances in the literature, often relying on vague epistemic recommendations. The article suggests that reasoning by analogy is a particular case within a broader framework of reasoning by analogy and by difference, shedding light on various analogical debates.

1. Introduction

Reasoning by analogy is a very common mode of reasoning both in everyday speech and in science, although it in no way guarantees the validity of the conclusions it proposes. For some authors, it even lays at the heart of cognition in general, allowing us at any time to draw on our past knowledge to cope with the present (Douglas Hofstadter, 2013).

Yet, it is well known that poorly constructed reasoning by analogy can lead to totally absurd conclusions. Some authors do consider that any analogical reasoning is a fallacy (Monroe Curtis Beardsley (1989)), merely rhetorical arguments (Susan Stebbing, 1939) or very fanciful comparisons (Jacques Bouveresse (1999)).

Then, the first questions that arise are the conditions under which analogical reasoning can be considered usable and the degree of reliability that can be attributed to it. A great deal of work has been devoted to trying to make explicit the rules that would make it possible to characterise robust reasoning by analogy as opposed to fanciful reasoning, following careful methodological protocols with many qualitative criteria, such as those proposed by John Maynard Keynes (1921), Marie Hesse (1966), Andrea Juthe (2005), or Paul Bartha (2010). But none of these protocols can vindicate the conclusion of analogical reasoning on purely logical or even probabilistic grounds.

Some of these works, such as Trudy Govier (1989) or William R. Brown (1989), insist on the duality of analogical reasoning, and the diversity of definitions and conditions proposed in the literature lead to complex taxonomies of analogies and theories on it, such as those of Juthe (2014, 2016), or Bartha (2010, 2013). So, the second question is whether analogical reasoning may be considered as a unique reasoning mode or on the contrary a very diverse one with many cases which cannot be unified within a single theoretical framework.

When analogical reasoning is considered as acceptable according to some contexts and conditions, the third question is whether it is a specific reasoning mode, a fourth one besides the classical trilogy deduction – induction – abduction, or is reducible to one of them, typically to deduction with missing premises or to general induction.

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The thesis supported by Walliser, Zwirn, Zwirn (2021) is that reasoning by analogy can be analyzed as a general type of inference scheme, whose validity in different logics depends on the background "determination rules" that are added to the reasoning. The paper extends this previous work and intends to answer to the three questions exposed along the following eight parts:

After having briefly recalled the notions of properties, relations, and functions that constitute the logical framework of our analysis, we present in detail the concept of "point of view", extending the concept of domains of properties to image sets of a function (§2). We then present the various kinds of determination rules, and generalize the deductive interpretation to a gradualist interpretation, including probabilistic and non-monotonic versions (§3). These notions are extended to other kinds of meta rules that may be in our background knowledge and which relate more generally similarities and differences to other similarities and differences (§4). Independently of these meta-rules, we define analogy as a notion between objects, always relative to a point of view and we show how a unified definition may be applied to prima facie various kinds of analogies, such as relational ones or even analogies between objects with different properties at a first level (§5). We then combine all these notions to expose the four logical inference schemes associated with reasoning by analogy and difference (§6). A special part is dedicated to the risks of redundancy and exceptions inherent to this reasoning (§7). All these elements allow to apply the general inference schemes to various cases of analogical debates, when first level knowledge or other analogies (competing analogies, rival analogies, additional analogies, or disanalogies) are introduced (§8). The conclusion is dedicated to compare our approach with some main other works and to test the solidity of the thesis of universality of our theory (§9).

The conclusions are those ones:

Reasoning by analogy can be rationale if one follows the general inference schemes proposed.

The strength and direction of the application of these inference schemes within different logics (deductive, non-monotonic, probabilistic) depends on the strength and direction of the meta-level rules that are added to the reasoning. But the inference schemes themselves are universal and can be applied to the diversity of cases raised by the analogical argumentation, or debates.

In this sense, analogical reasoning is not a new consequence relation or a new logic. It simply follows the same logic as those used to express the meta-level rules on which it relies. Nor does it represent a fourth mode of reasoning besides the usual deduction-induction-abduction trilogy. The inference schemes presented may be applied to single case inductions, even if concrete examples are often different.

If the conclusion of any reasoning by analogy inherits the epistemic strength of the determination rules on which it relies, it is nevertheless most of the time a defeasible reasoning, open to debates, criticisms, or on the contrary reinforcement.

2. Functions, properties, and points of view

We will use without restriction higher order logic and especially second-order logic, which is the only logic powerful enough to express the concepts that are needed to analyze the reasoning by analogy. It is well known that contrary to the logic of predicates, these logics suffer from problems on the mathematical level. They satisfy neither the completeness theorem nor the compactness theorem. But these flaws, which are important in formal logic, are of little concern for our present purposes. We will therefore use them without further concern.

Let us consider a set **O** of objects A, B, C..., X denoting any object of this set. They can be specific concrete objects like an apple, generic objects like the set of French people, or conceptual objects like the set of even numbers.

These objects have properties which are formally represented by a set **Π** of predicates P, Q, ... etc. Predicates are used to formulate statements such as P(X), which is true if and only if the object X satisfies the property P. To each property, we can associate the set [P] of the objects which satisfy it. A predicate can thus be seen either as an element of the set **Π** of predicates on **O** (in which case we write P ∈ **Π**) or as an element of $\mathscr{P}$**(O)** (the set of subsets of **O**), which we label [P]. These properties, and the associated sets of objects, can correspond to qualities or classes usual or not in ordinary language: mammals, red things, objects 53 centimeters long, etc.

Relations R, S... between two or more objects, represented by predicates of any arity n, are more generally used to form statements such as R (X, Y... Z), which are true if the n-uple (X, Y... Z) satisfies the relation R. To each of these relations one can associate the n-uples which satisfy them, for example, the set of triplets of integers whose first one is the square of the second one and the second one is the square of the third one.

The properties – or relations – can also have second-order properties – or relations – which apply to the first-order properties. By abuse of language, we will call “generic objects” the set of objects that satisfy a certain predicate. This applies, for example, to the set of objects satisfying the predicate "to be an apple" which can be generically called "the set of apples" or in a shortened way "apples". We can then say for example that the predicate abundant(apples) applies to the generic object "apples" and that this statement is true if there are many apples, which is a way of attributing a second-order property to the predicate "to be an apple". We will then use the notation **P** instead of [P] to remind us that we consider **P** as an object to which higher-order predicates can also be applied. We will see that the use of generic objects simplifies the fact of formulating a universal representation of analogy considered as a relation between objects (§5).

Generic objects have properties that do not necessarily make sense at the level of individual objects. If apples are abundant in winter, we cannot say the same thing about this particular apple. Similarly, if red is a warm color, we cannot say that this rose, red, is itself a warm color. Even if it seems in common language that we can say that this rose "has a warm color", we are in fact expressing a double idea, a first-order idea: this rose has a certain color (not stated) and a second order idea: this color is a warm color.

All objects have many properties because they can all be included in a large number of sets with a list of other objects, even if not all those sets are considered as properties in our current language and classifications, as for example, the animals "drawn with a very fine brush made of camel hair", or "that just broke the jug" mentioned in a text by Borges quoted by Michel Foucault (1966). Nothing in predicate logic prevents each object from having different properties simultaneously: the propositions P(A) and Q(A) are not *a priori* logically contradictory if P and Q are any two predicates. Of course, it may be that, empirically, we know that the extension of P&Q is empty: no Frenchman is richer than Bernard Arnault in 2023. But this is not a question of logic or language, only a question of (social) fact.

Ludwig Wittgenstein (1930) argues, on the contrary, that certain simple propositions, such as P(A) and Q(A), are not *a priori* independent and can contradict each other without depending on an *a posteriori* empirical statement. Common sense itself forbids statements like "A is green and A is red". Current logical reasoning can have the form “a man is 2 meters tall, therefore he is not 3”: there is no need to measure this man to accept this. These constraints correspond to “categories”, within which two values, “two determinations of the same kind are impossible”. Thus, according to Bruno Leclercq

(2018, p. 107), “the idea of a new ‘philosophical grammar’ is born, which is concerned with highlighting a series of non-logical categorical concepts – such as that of color.

To represent these constraints, which are not empirical truths but are related to language itself, one can gather predicates into “domains”, allowing one to express the constraints associated with these categories of Wittgenstein. Domains are sets of predicates that are incompatible two by two: contrary to the general case, it is not possible for the same object to satisfy two of them simultaneously. Thus, the domain of colors will include for example all the colors of the chromatic circle, the domain of three-dimensional shapes the list of different objects like cube, sphere, pyramid, cylinder, etc. The list of domains, as well as their content, is given exogenously but supposedly shared by competent speakers of the same language. We will say that a domain, denoted Z, is a set of properties (and therefore of predicates) that are incompatible by pairs.

Employing domains as such consists in considering objects from a certain "point of view". The point of view of their color, their size, their nationality, etc., among the infinite characteristics they may possess.

A statement of the type P(X) is false if X does not satisfy the property P, which can correspond to two different situations: either the object satisfies a contrary property or the predicate P does not "apply" to this object, a notion that has no formal translation in predicate logic. These two situations correspond respectively to the following examples: "this bird is blue" is false because this bird is green, but "the number 5 is blue" is false because a number has no color. We will then say that a domain Z is "relevant" for an object X if there is a property P of Z such that P(X) is true. This object will be "concerned" by this domain.

The concept of domain can be extended to functions, in a rather natural way. As it was done for predicates, we can describe the concept of property satisfaction by a set of functions (noted **Ψ)** f, g, h... from objects of **O** or n-uples of objects of **O** to a set of values **V**. Having a certain value for a function is the same as having a certain property, and conversely satisfying a certain property can be expressed by having a certain value for a function. The value of this variable corresponds to what Wittgenstein calls “the determination of a species” or “the specification of a category”: weight, size, color, etc. The list of possible values is not given by an extensional definition internal to logic but by the very meaning of these categories that fixes the way we describe the world. Their characteristic is that the same object cannot have two different values (in the same place, at the same time). It is as if we were using evaluation protocols that can only have one result which is the attribute of a functional value to the object being measured. For example, Red (X) can be translated by a function f which associates the “red” value to X in a set of possible colors. The correspondence between functions and predicates is then expressed in the following way:

For any function f $\in\Psi$, for any value α, we can define a predicate $P_{f\alpha}$ such that:

$[P_{f\alpha}] = \{X, f(X) = \alpha\}$ or equivalently $P_{f\alpha}(X)$ if $f(X)= \alpha$

Then we see that setting a function f from the set of objects to an image set of values is equivalent to setting a domain. Each predicate in this domain is equivalent to the property $f(X)= \alpha$ for one of the possible values of the image set of f. As a result, the pairwise incompatibility between the domain predicates (a fundamental condition of the definition of what we have called a domain) is automatically verified. The concept of a generic object translates into:

$\mathbf{F_\alpha} = [P_{f\alpha}] = \{X, f(X) = \alpha\}$

All these definitions can be generalized to functions with multiple arguments.

These two modes of expression, predicative and functional, are interchangeable but may lead to more or less easy ways to express things depending on the context. In particular, the functional expression is shorter to handle for representing determination rules (§4.) and analogical debates that rely on them (§8.).

3. Determination rules

When we consider two domains Z and Z', there may be some relations between the properties of these domains that represent a form of influence or causality, i.e., a determination of the properties of one domain by those of the other. Whether this determination can be thought to be causal or purely epistemic is not a controversial question for our present needs because we are interested by the belief of the agent, which can include belief about causes. These relations are expressed by "determination rules", which are meta-level relations that link sets of properties and are part of many common beliefs such as:

- *Nationality determines an individual's native language.*
- *The species of an animal determines its diet mix or its color.*
- *The socio-professional category of a citizen determines her vote.*

They can also concern relationships, such as:

- *The type of family relationship determines how often two people meet.*
- *The distance between two masses determines the intensity of their mutual attraction.*

The contents of domains can be considered, following Wittgenstein, as rules linked to our "language games" independently of any empirical data: the fact that an object is either blue or yellow but not both is not contingent, what we mean in general by blue is incompatible with what we mean in general by yellow. But the determination rules that link two domains are purely empirical. They express the contingent fact that certain similarities, in one domain, create other similarities, in other domains. Nelson Goodman (1947, pp.108–09) gives a method of theoretical construction of this concept, in the context of his analysis of projectibility.

Formally, the fact that the domain Z *strictly determines* the domain Z' can be stated as follows:

$$Z \blacktriangleright Z' \text{ iff } \forall P \in Z, \forall Q \in Z', [\forall X\, (P(X) \rightarrow Q(X))] \vee [\forall X\, (P(X) \rightarrow -Q(X))$$

Or equivalently:

$$Z \blacktriangleright Z' \text{ iff } \forall P \in Z, \exists Q \in Z', [\forall X\, (P(X) \rightarrow Q(X))]$$

And in functional terms:

$$f \blacktriangleright g \text{ iff } \forall X, \forall Y, [f(X) = f(Y)] \rightarrow [g(X) = g(Y)]$$

or equivalently:

$$f \blacktriangleright g \text{ iff } \forall \alpha, \exists \beta, \forall X\, [f(X) = \alpha \rightarrow g(X) = \beta]$$

A determination rule expresses the fact that, for two domains or two functions, the similarity between two objects creates some additional similarity. The functional formulation will be more often privileged hereafter to analyze the determination rules because of its greater convenience, but all the proposed definitions have a predicative equivalent.

It is very rare to believe in strict determination rules for empirical properties. Such rules, while common in our background beliefs, are almost always fraught with exceptions, such as:

- *Nationality determines an individual's native language – unless the individual is a naturalized immigrant.*
- *The species of an animal determines its diet mix – except for newborns.*
- *The socio-professional category determines the vote – but age, city or religion may lead to deviating from this determination.*

This situation of lower belief is therefore better represented by the following form:

$f \rhd g$ iff $\forall X\ \forall Y, [f(X) = f(Y) \hookrightarrow g(X) = g(Y)]$

or again:

$f \rhd g$ iff $\forall \alpha, \exists \beta, \forall X, [f(X) = \alpha \hookrightarrow g(X) = \beta]$

where the arrow $\hookrightarrow$ is a non-monotonic inference relation, as defined for instance by the logic of preferential models (Sarit Kraus, Daniel Lehmann, Menachem Magidor, 1990, or "KLM"), or any other non-monotonic logic of the same kind. The precise axioms satisfied by this logic are not important here if they allow the construction of a coherent inference relation that can be undone by adding new information to the premises. Those formulas represent therefore non-monotonic determination rules in which having a certain value according to the function f non-monotonically implies having another definite value according to the function g.

An6extension of the logical framework formulated by KLM to statements of functional or predicative type is not required for the relevance of the present paper: functional statements such as f(X) = f(Y), g(X) = g(Y) can be treated as propositional statements p, q, that denote them insofar as it is not their content that is the object of a logical analysis here.

But it is also possible that these beliefs have a purely probabilistic nature, when the first domain (f) is only one of the determinants – causes or explanations- of the second domain (g).

Situations in which one believes, without certainty, that it is possible that $f \blacktriangleright g$, can be represented by a probabilistic degree of belief: $Pr(f \blacktriangleright g) = \lambda$. But in the most common instances where one firmly believes that the determination of one domain is only one determination among many, the previous representation falls short. If one believes that $f \blacktriangleright g$ is false (since there are exceptions), then one does not hold the belief that $Pr(f \blacktriangleright g) = \lambda > 0$. This formulation, proposed in Walliser, Zwirn, Zwirn (2021) to represent reasoning by probabilistic analogy, does not adequately convey an agent's belief in these most common situations where the agent knows that the determination rule is not true in a strict sense because of possible exceptions. What is meant here is not that one has a subjective degree of belief in a strict determination rule, but the fact that the very nature of this rule is probabilistic.

Hence it must be expressed by:

$f \rhd_{\lambda} g$ iff $\forall X, \forall Y, [Pr((g(X) = g(Y)) / (f(X) = f(Y)))] = \lambda$

This formula represents a probabilistic determination rule in which having a certain value according to the function f makes it probable with a degree λ to have another very precise value according to the function g. This corresponds rather to the third example:

- *The socio-professional category generally determines voting quite well – but age, city, and religion are also important determinants. This is demonstrated by statistical data from post-election surveys.*

The fact that $f \triangleright_{\lambda} g$ provides a gradualist indication of what the agent believes to be the objective determination of g by f. Of course, this does not exhaust all possible measures of uncertainty. Other methods, such as probability intervals or Dempster-Shafer belief functions which can be analysed in the context of belief hierarchies (Walliser & Zwirn, 2011) could be used. But this is general enough for the present analysis of the way determination rules and analogical reasoning run.

When $\lambda > S$ (where S is large enough) it can be stated that similarity according to f "renders plausible," or "absolutely confirms" similarity according to g. In the limit, if λ is as near to 1 as desired (though not strictly equal to 1), $f \triangleright_{\lambda} g \Leftrightarrow f \triangleright g$. For further insight into this, refer to Judea Pearl's 1988 interpretation of Adams' probabilistic semantics in the context of non-monotonic logic.

If $\forall X, \forall Y$, [Pr((g(X) = g(Y)) / (f(X) = f(Y)))] > [Pr(g(X) = g(Y))], then similarity according to f supports or "relatively confirms" similarity according to g. This highlights the distinction between absolute and relative confirmation, as discussed by Zwirn & Zwirn (1996).

Determination rules are practically employed to suggest an explanatory or causal mechanism: similarity according to f plays a cognitive role on similarity according to g.. But in situations where $\forall X$, $\forall Y$, Pr(g(X) = g(Y)) is already significantly large, the use of the determination rule of g by f seems to be irrelevant. Such a situation is illustrated in the forthcoming example:

- *Given that 90% of French people believe that the Earth is round, the probability that two French people share this belief is 0.81. The fact that two people weigh the same or not does not influence this probability. Therefore, the chance that two French individuals believe that the Earth is round is 0.81 when they weigh the same which is quite high. So, in this sense, one could argue that their weight somewhat determines French people's belief about the roundness of the Earth. However, this conclusion is misleading due to the lack of a true causal or correlational relationship between the two factors.*

Finally, one can consolidate the three kinds of determination rules into the gradualist form $f \triangleright_{\lambda} g$ by noting the following special cases:

$f \triangleright_{\lambda} g \Leftrightarrow f \blacktriangleright g$ when $\lambda = 1$

$f \triangleright_{\lambda} g \Leftrightarrow f \triangleright g$ when $\lambda = 1-\varepsilon$, with $\varepsilon > 0$ as close to 0 as one would like

While it remains a philosophical debate whether P(Y/X) = 1 is equivalent to $X \rightarrow Y$, it doesn't impact our current exploration. Notably, if one believes that $X \rightarrow Y$, then one also believes that P(Y/X) = 1. This is relevant enough here when replacing Y and X with g(X) = g(Y) and f(X) = f(Y), respectively. In this case, the value of g is certainly determined by the value of f.

The fact that P(Y/X) = 1-ε, with ε > 0 as close to 0 as desired is equivalent to $X \hookrightarrow Y$ was proven by Judea Pearl (1988). This provides an interpretation of Adams 'probabilistic semantics in terms of non-monotonic logic.

4. Other meta-level rules

Determination rules are not the sole meta-level rules that can connect two domains.

Indeed, it is also possible that dissimilarity within one domain corresponds with dissimilarity in another domain, with or without exceptions, just as with the determination rules. Consider the following examples:

- *Inhabitants of different European countries typically speak different native languages.*
- *Children of various ages are often at different school levels.*

Let's refer to these kind of rules as "separation rules". Unlike determination rules, separation rules articulate that dissimilarity creates dissimilarity, which is not implied by the fact that similarity breeds similarity. This can be written in the following manner using the functional notation:

$$f /_{\lambda}\ g \text{ iff } \forall X, \forall Y, [Pr((g(X) \neq g(Y)) / (f(X) \neq f(Y)))] = \lambda$$

As previously, we have the equivalence:

$f /_{\lambda}\ g \Leftrightarrow \forall X, \forall Y, [Pr((g(X) \neq g(Y)) \rightarrow (f(X) \neq f(Y)))]$ when $\lambda = 1$ (strict separation rule)

$f /_{\lambda}\ g \Leftrightarrow \forall X, \forall Y, [Pr((g(X) \neq g(Y)) \hookrightarrow (f(X) \neq f(Y)))]$ when λ is as close to 1 as one would like (non monotonic separation rule)

When determination rules and separation rules are combined, we get "decisive rules". These rules establish that any similarity from a given point of view implies a similarity from another point of view, and the same with differences. That means that only similarity can generate similarity.

A determination rule is thus decisive if it leads to the existence of an injective application between the set of values of f and the set of values of g. As an example:

- *The city of residence determines the mayor who governs the city, and if two people live in two different cities, they undoubtedly have two different mayors.*

As mixed rules, decisive rules can be strict, non monotonic, or purely probabilistic in one respect (e.g., the determination rule) and strict, non monotonic, or purely probabilistic in another (e.g., the separation rules).

Decisive rules might not be the most common case. For instance, the species of an animal often determines quite well the color of an adult of that species, yet no one claims that only crows are black. Nevertheless, such rules form a part of our prevalent beliefs in some domains.

Lastly, there are also less intuitive "counter-determination" scenarios. In these situations, the fact that two objects share a specific property implies that they differ in another property: similarity leads to a difference. In this case, similarity generates dissimilarity. This is written as follows in probabilistic terms:

$$f \lhd_{\lambda} g \text{ iff } \forall X, \forall Y, [Pr((g(X) \neq g(Y)) / (f(X) = f(Y)))] = \lambda$$

As an example:

- *Sonship counter-determines paternity: having the same son implies not having the same father (with the pharaonic exception)*

Counter-determination can either be "total", represented by the previous formulas, or "partial". A partial counter-determination concerns only certain values of the function f. This can be stated as follows:

$f \lhd_\lambda g$ iff $\exists\alpha, \forall X, \forall Y, [Pr((g(X) \neq g(Y)) / (f(X) = f(Y) = \alpha))] = \lambda$

As an example:

- *Let's consider a 2-valued function f: f(x) = 1 if x was a candidate in the last municipal election in Paris, f(x) = 0 otherwise. Let g be a function such that g(x) = β if x voted for β in the same election. In this scenario, the fact that two persons were candidates in this election strictly counter determines the fact that they voted for the same candidate. Conversely, for individuals who were not candidate, the identity of f(x) has neither a positive nor a negative influence on voting.*

Symmetrically, one could have "counter-separations" rule, when dissimilarity creates similarity, which is written as follows:

$f \setminus_\lambda g$ iff $\forall X, \forall Y, [Pr((g(X)= g(Y)) / (f(X) \neq f(Y)))] = \lambda$

We can consider an example that is the reverse of the previous one:

- *Difference in paternity counter-separates sonship: not having the same parents implies having a chance to have the same son.*

Admittedly, these types of rules may appear even more unusual than counter-determination rules. However, the provided example demonstrates that it's at least possible to get such rules in the background knowledge, even if they're associated with a low probability. At the bare minimum, not having the same parents enables a higher probability of having the same son than having the same parents. This occurs with a higher probability than in general (i.e., without knowing if the two people have the same parents or not).

So, to project the property of one object onto another one can use the four logically possible types of meta-level rules linking two domains of properties:

- Similarity may favor other similarities (determination rules)
- Dissimilarity might encourage further dissimilarities (separation rules)
- Similarity could support some dissimilarities (counter determination rules)
- Dissimilarity may lead to some similarities (counter separation rules)

The first two rules are the more commonly represented in our background knowledge, while the latter two are more exceptional. Only the first one has been identified in previous literature (though not completely analysed) and is associated with analogical reasoning (see §6.). However, all four kinds of rules can be utilized in broader patterns of reasoning by similarity and difference and applied to various situations of analogical debates (see §8.)

5. Analogies

Two objects can share the same property within a domain Z or have identical value for a function f whose possible values correspond to the properties of Z. These objects can then be said to be "similar" or "analogous" respectively to this domain or function:

$A \sim_Z B$ iff $\exists P \in Z, P(A) \& P(B)$ or equivalently $A \sim_f B$ iff $f(A) = f(B)$

Analogies are only meaningful when considered in relation to a domain or a function: there exists no absolute analogy or similarity. This point was highlighted by Walliser, Zwirn, Zwirn (2021), building upon the observations of Willard Van Ornam Quine (1969). Despite that analogy is generally expressed in an asymmetric way with a "source" B and a "target" A, this definition makes analogy an equivalence relation between objects. The contrast between source and target only become significant pragmatically when using analogies for reasoning. This concept can be expanded to analogies amongst n-uples of objects sharing an n-ary relation R within a domain Z of mutually exclusive n-ary relations:

$(X,\ldots Z) \sim_Z (X',\ldots Z')$ iff $\exists R \in Z, R(X\ldots Z) \& R(X'\ldots Z')$

This approach allows to represent "proportional analogies" as a special case. These expresses the concept that "A is to B what C is to D" or the equations that link several variables in distinct domains by the same relation, such as those found in physics or economics. These proportional analogies can combine objects of different levels (individual, generic), and express shared relations between them. They further extend the expression of other forms of relativity of analogies without altering the general definition. For example, the fact that succeeding in one's professional life is like winning a sport competition, from the point of view of the happiness it brings, is not true for everyone, but it does for Ana. This does not necessitate proposing a new definition of analogy relative to individuals but can be conveyed in the common form: succeeding in one's professional life is to Ana what winning a sport competition is to her, in terms of the happiness it brings her.

The relationship between objects could be extended to properties. It is then possible to consider first-order predicates as generic objects:

$\mathbf{P} \sim_Z \mathbf{Q}$ iff $\exists T \in Z, T(\mathbf{P}) \& T(\mathbf{Q})$

Here, **P** and **Q** represent the generic objects correlated to the populations of first-order objects that satisfy the predicates P and Q. T is a second order predicate applicable to these generic objects within a set Z of disjoint predicates that can be applied to them. Therefore, we can say that apples are like pears because they are abundant types of fruit. This is equivalent to considering the corresponding predicates Apple and Pear as analogous, relatively to the domain of abundance/scarcity. Utilizing the concept of analogy between properties in this manner preserves the universality of the definition.

The existence of an analogy between two distinct properties allows us to regard as similar two objects that might not have been otherwise considered as such. In this situation, we can say that the diverse properties of the two objects "correspond" with each other at a different level. Consequently, these properties could form the basis of an analogy between the objects.

To illustrate this, we could consider a box A analogous to a box B, despite the fact that box A is a cube and box B a pyramid. This is because both shapes are angular, unlike spheres for instance. This relationship is symbolized by the predicates "cube" and "pyramid" both exhibiting the second-order property of being “angular” shapes. So they are analogous properties. If we extend the definition to state that a box itself (not just its shape) can be angular, it's equivalent to saying for any X, if cube(X) then angular(X), and if pyramid(X) then angular(X). From the shape aspect, these two boxes are not analogous, but they become analogous when considered from the perspective of the type of their shape, which is a broader domain, or “point of view”.

This is formally expressed in the following way: Let A, B, such that $A \sim_Z B$ is false because this is the case that $P \in Z$, $Q \in Z$, $P \neq Q$, P(A), Q(B).

Let Z^ be a domain of predicates that applies to generic objects associated with predicates of Z and such that $\forall P \in Z, \exists T \in Z\text{^}, T(\mathbf{P})$.

Suppose $\exists T \in Z\text{^}$, such that T(P) and T(Q), so $\mathbf{P} \sim_{Z\text{^}} \mathbf{Q}$

Given $T \in Z\text{^}$ define ϕ_T as the following predicate: $\phi_T(X)$ iff $\exists P \in Z, P(X)$ et $T(\mathbf{P})$

Then let Z+ be the set of all predicates ϕ_T : Z+ = { ϕ_{T}, $P \in Z$, $T \in Z\text{^}$}. This set is indeed a domain: $\forall X$, $\phi_T \neq \phi_{T'} \Rightarrow [\ \phi_T (X) \Rightarrow -\ \phi_{T'} (X)]$. Indeed, suppose that ϕ_T (X) and $\phi_{T'}$ (X). So $\exists P \in Z$, P(X), $T(\mathbf{P})$ et $\exists P' \in Z$, P’(X), $T'(\mathbf{P'})$. Since Z and Z^ are domains: P = P' and T = T', so $\phi_T = \phi_{T'}$.

Then $A \sim_{Z+} B$ because ϕ_T (A) and ϕ_T (B) and $\phi_T \in Z+$. Indeed: $P(A) \Rightarrow \phi_T (A)$ because $T(\mathbf{P})$ and $Q(B) \Rightarrow \phi_T$ (B) because $T(\mathbf{Q})$.

The key point is the assumption of the analogy between P and Q. This allows in all cases to derive the analogy between A and B, relatively to a domain Z+ determined by the domain Z^ that grounds the analogy between these properties.

However, finding an analogy between two properties does not automatically qualify the objects to which these properties apply as analogous. It only works if one is willing to adjust one’s perspective, seeing the properties in question as relevant for comparing the objects or justifying the reasoning based on this analogy. Take, for example, other properties of the shapes mentioned earlier: suppose cubes and spheres are classified as "simple" shapes, while cylinders and pyramids are "complex" shapes. Imagine two boxes, one cube-shaped and the other sphere-shaped, which wouldn't typically be considered analogous due to their contrasting shapes - angular vs. rounded. But if one alters one’s perspective to view the cube and sphere as analogous because both are simple shapes, these two boxes can then be seen as analogous, but only if we are open to comparing them in terms of the simplicity or complexity of their shapes - a domain that may or may not be useful for subsequent reasoning.

6. Reasoning by similarities and differences

So far, we have defined domains and their properties. We have also developed meta-level rules that connect them and have established analogies with respect to these domains. However, we have yet to apply them to reasoning by analogy. To take this additional step, we must build upon all these prior definitions.

Reasoning by analogy consists in relying on an analogy to project a property of an object onto another. This may be stated in the predicative or functional forms as follows:

If $A \sim_Z B$, and Q(B), then Q(A)

If $A \sim_f B$ and $g(B) = \beta$, then $g(A) = \beta$

The challenge lies in justifying the rationality of such common and sometimes intuitive reasoning. Its premises do not inherently guarantee its conclusion, so it is an "ampliative" form of reasoning. As such, the purpose cannot be to assert its validity outright, because it definitively is not valid. Instead, the aim is to provide an explanation for why it seems acceptable to form coherent beliefs about its conclusion, given some new premises.

The approach we suggest, initially presented in Todd Davies & Stuart Russel (1987) and subsequently refined in Walliser, Zwirn, Zwirn (2021), posits that reasoning by analogy necessitates an overarching meta-level background hypothesis, which validates it as an inference scheme across various logic types. The nature and degree of belief in the validity of the conclusion is contingent upon the type of logic applied for the inference and the nature and degree of belief in the background hypothesis. This strategy does not call for a new logic rooted in a purportedly "analogical consequence relation", but rather aims to pinpoint a specific model of inference scheme. This model permits connections between the background hypothesis and conclusion, following diverse types of consequence relations.

The type of background hypothesis used within this inference pattern model is a determination rule which connects two domains or two functions. Indeed, one can complete the prima facie reasoning by analogy according to the following inference scheme model in the functional formulation:

$[A \sim_f B,\ g(B) = \beta,\ f \rhd_\lambda g] \Rightarrow_\lambda g(A) = \beta$
Or:
$[f(A) = f(B),\ g(B) = \beta,\ f \rhd_\lambda g] \Rightarrow_\lambda g(A) = \beta$

Arrow $\Rightarrow_\pi$ is a probabilistic inference relation, defined for example by Carl Hempel (1965) and such that: $[(P(B/A) = \pi) \ \&\ A] \Rightarrow_\pi B$ expresses the fact that, in the absence of new evidence to the contrary, the strength of the inference towards a proposition is equal to its conditional probability with respect to the premise. It is a defeasible relation because the following property is false: $[(P(B/A) = \pi) \ \&\ A \ \&\ C] \Rightarrow_\pi B$.

As seen above, $f \rhd_\lambda g$ is equivalent to $f \blacktriangleright g$ when $\lambda = 1$ and equivalent to $f \rhd g$ when λ is as close to 1 as one wants.
Hence, in these cases, the preceding formula is equivalent to these ones:

$[f(A) = f(B),\ g(B) = \beta,\ f \blacktriangleright g] \Rightarrow g(A) = \beta$

$[f(A) = f(B),\ g(B) = \beta,\ f \rhd g] \hookrightarrow g(A) = \beta$

The arrow $\Rightarrow$ is a deductive consequence relation. The arrow $\hookrightarrow$ is a non-monotonic consequence relation, identical to the one used in the definition of $f \rhd g$.

It is easy to verify that when $f \rhd_\lambda g$, $f \blacktriangleright g$, $f \rhd g$, are defined as above, all the above inference schemes are valid. For example, in the probabilistic case if one replaces $f \rhd_\lambda g$ by its definition:

$[f(A) = f(B),\ g(B) = \beta,\ [\forall X, \Pr((g(X) = g(Y)) / (f(X) = f(Y))) = \lambda] \Rightarrow_\lambda g(A) = \beta$

For the sake of simplicity, we will use only this general probabilistic inference scheme in the following, with $0 \leq \lambda \leq 1$.

Depending on the nature of the determination rule employed by the agent the conclusion of reasoning by analogy can range from certainty to acceptance and to varying degrees of probability. The conclusion is backed by reasoning by analogy, without necessarily being accepted, when its posterior probability surpasses its prior probability. As demonstrated above, a conclusion with high probability is typically only associated with a strong support; this is because, in empirical situations, the likelihood of any two objects sharing the same value assignment for a given function is ordinarily quite low. In an edge case where this is not so, reasoning by probabilistic analogy results in high probability of the conclusion maintaining its validity but losing its force of assertion or relevance. This seems odd as it doesn't contribute any new knowledge.

It is easy to check that the logical form of reasoning by analogy is strictly identical to that of induction based on a single case. We can conclude that reasoning by analogy is not a fourth mode of reasoning, but the general form of the first stage of any induction. The difference between typical cases of one-case induction and typical cases of reasoning by analogy is purely pragmatic and grammatical, as discussed in Walliser, Zwirn, Zwirn (2022).

Analogical reasoning relies on determination rules to derive new analogies from a previous one. But, as detailed in §5, other kind of meta-rules may link two domains and lead to other general schemes of reasoning.

If one starts from an analogy, one may sometimes derive disanalogies, if the two domains are related by a counter-determination rule. The general inference scheme is the following:

$[A \sim_f B, g(B) = \beta, f \lhd_\lambda g] \Rightarrow_\lambda g(A) \neq \beta$

If one starts from a disanalogy, one may also derive disanalogies, if the two domains are related by a separation rule, along the following inference scheme:

$[\text{NOT } A \sim_f B, g(B) = \beta, f /_\lambda g] \Rightarrow_\lambda g(A) \neq \beta$

On the contrary, if one starts from a disanalogy, one may derive analogies, if the two domains are related by a counter-separation rule, along the following inference scheme:

$[\text{NOT } A \sim_f B, g(B) = \beta, f \backslash_\lambda g] \Rightarrow_\lambda g(A) = \beta$

It is easy to check that these three schemes are valid if one replaces $f \lhd_\lambda g$, $f /_\lambda g$, $f /_\lambda g$ by the definitions given in §4.

As noted above, the arrow $\Rightarrow_\lambda$ denotes a probabilistic inference, assigning the conclusion with a probability $0 \leq \lambda \leq 1$. In all four inference schemes, λ may be very weak and yet still support the conclusion if it surpasses the a-priori probably of the conclusion, as in the different parents / same son example. λ may be as close to 1 as desired and $\Rightarrow_\lambda$ will be equivalent to a non monotonic inference, or equal to 1, and the conclusion will be as certain as if it were a deductive relation.

Analogical reasoning merely represents one of several approaches to reasoning based on similarities and differences, even if it appears to be the most common. In all four instances, the conclusion is logically derived from a unique general inference scheme and does not necessitate any additional conditions, often associated with such reasoning in the literature on analogical reasoning.

In instances where no meta-level rule is known or put forth by the agent, another party can either contend that her reasoning holds no epistemic value or bring forth one meta-level rule to dispute the validity of the conclusion drawn from the reasoning. This addresses Bartha's (2013) arguments against using determination rules for proposing a universal analysis of analogical reasoning. Even in this case, they are ruling the issue of the reasoning. In other words, when one performs this kind of reasoning, all things are as if there were a criterion of “epistemic commitment”: one is committed to believe, with degree λ, to the meta-level rule that is logically necessary to draw the conclusions one draws with degree λ.

The remaining of the paper will be focused on analogical reasoning only, but we will show the possible use of other similarity and differences forms of reasoning when considering analogical debates.

7. Redundancy and exceptions

Determination rules are meta-level empirical regularities, different from first-order empirical regularities linking two properties such as:

- Australians speak English.
- Lions are carnivores.
- Young students vote Left.

It was Davies & Russel (1987) who initially highlighted that choosing determination rules, as the underlying hypotheses in reasoning by analogy, circumvents the potential pitfall of “redundancy”. This redundancy risk diminishes reasoning by analogy to merely applying a first-order regularity to the target object. In scenarios where one knows or believes that a property P inherently implies another property Q, there is no need to observe a second object B to determine that object A possesses property Q if it has property P. Reasoning by analogy becomes superfluous in this case. While it can still be conducted, its assertion is trivial. It is superseded by a direct inference based on first-order generic evidence; a process that can be termed “focusing” on the target object, negating the need for information on the source object.

However, a determination rule could be seen as a partition that correlates each property of a domain Z with a property of a domain Z'. Consider the example where an animal's species determines its dietary bundle (defined as the set of diets it can have). This would be interpreted as follows:

- Lions are carnivores.
- Cows are herbivores.
- Birds are granivores, insectivores and fructivores.
- And so on for all animal species.

In other words, even though a determination rule operates at the meta-level, it essentially constitutes a list of first-level regularities. The concept of a meta-level merely serves as a summary that could invariably be projected to the first level through an expanded list. Consequently, it should not represent a distinct type of knowledge.

This interpretation, however, misses the crux of the matter by assuming that one knows all the first level rules or at least has beliefs regarding all of them. Naturally, believing in a determination rule requires acceptance of the notion that such a partition exists in the real world, which could be partially or entirely discovered. But the usefulness of analogies comes into play when it is not yet the case.

For instance:

- *When one believes that a type of animal species determines its dietary mix, there is a belief regarding a general causal link (either strict or flexible) between the two types of properties. The belief is that an animal's diet is typically dictated by its species, regardless of specific species involved. This is a meta-level connection. It could align with other beliefs about the relationships between an animal's organs, predatory abilities, natural environment, etc., with its diet composition. However, we often lack knowledge about the diets of many animals, such as the dietary traits of the platypus.*

- *We generally assume that the country of birth determines quite well the spoken language due to obvious institutional and sociological reasons, but we often don't have information about the most spoken language in each country, say Suriname. But if one knows that Willem, a resident of Suriname, speaks Dutch, and if one also knows that another individual, Saskia, lives in Suriname, it will lead one to surmise that Saskia likely speaks Dutch.*

Reasoning by analogy proves useful only in the absence of direct generic evidence pertaining to the target object. It comes into play specifically when the associated determination rule can't be fully reduced to first order regularities and when no first order regularity is applied to the case being examined. If we know or believe that native Surinamese individuals have a very low likelihood of speaking French, we can directly conclude that Saskia, born in Suriname, probably does not speak French either. There's no need to compare Saskia to Willem, another native Surinamese who does not speak French. The meta-level rule stating that the birthplace determines the language spoken becomes unnecessary.

However, encountering exceptions to first-order regularities, the situation can turn more complex. Take the previous example: suppose that Willem, a native Surinamese speaks French as his parents are French. If we employ country-of-birth analogy reasoning between Saskia and Willem, based on a rule where the country of birth determines the spoken language, we could potentially arrive at a false conclusion: that Saskia speaks French. Put simply, reasoning by analogy is inherently sensible to exceptions, transferring these exceptions from one case to another, indicating a significant structural flaw. This risk is consistently present in all cases of non-redundant reasoning by analogy, i.e., where the underlying regularity associated with the treated case is unknown. There will always be a risk of observing an anomaly, like a green raven, and extending this observation to another raven. Blocking this type of conclusion is only possible when the relevant first-order regularity is known. In Saskia's case, knowing that in Suriname people generally do not speak French overrides the need for reasoning by analogy from a special and potentially anormal case.

8. Analogical debates

When one performs a non monotonic or a probabilistic reasoning one ought to consider all the evidence at one's disposal. This is what is stated by the Requirement of Total Evidence (RTE), largely explored and debated in the philosophical literature since Rudolph Carnap (1950), which regards it as belonging to the methodology of induction. For instance, Paul Drapper (2020) defines it like this: "when assessing the credibility of hypotheses, we should endeavor to take in account all of the relevant evidence at our disposal instead of just some proper part of that evidence". Even though RTE can be disputed (see e.g., Peter Fisher Epstein, 2017) and if the concept of "relevant evidence" remains an open question, we will endorse it for our present needs.

Adhering to RTE doesn't imply that an inference scheme like the one we presented for analogical reasoning becomes invalid if we haven't collected all possible relevant evidence about the source and target object. RTE is a heuristic principle rather than a logical one: it encourages us to gather all pertinent information to secure our argument, but logic pertains only to the information that has been effectively collected.

For example, non-monotonic reasoning suggests that if p is given, and if q can be inferred monotonically from p, then q is a reasonable conclusion. However, it's always possible to acquire new evidence r such that in the presence of r, q is ultimately defeated: if we discover that r is true, we should discard q. This doesn't invalidate the initial reasoning if we do not uncover r or even do not seek to verify r.

Because reasoning by analogy is defeasible in the most common cases where the determination rule is not deductive, it is also open to debates that may challenge its conclusion each time one new relevant evidence is added. However, this new evidence does not change the logical validity of the general inference scheme, nor does it introduce any new condition for analogical reasoning. It simply suggests applying this inference scheme, and logic in general, based on the information acquired at a given time T. Notably, the other similarity and difference inference schemes can also be used in many typical situations described in the literature of analogy. This method circumvents the endless search for conditions that might render analogical reasoning more or less acceptable, with the impossible goal of listing a priori all possible situations which could support or defeat it in advance (John D. Norton, forthcoming).

Indeed, the types of analogical debates detailed below do not provide any additional criteria to this reasoning: they simply provide typical examples of the application of the same reasoning, as showcased in §6, applied to varying data and situations. The analysis of analogical debates is only a practical exercise, not a theoretical one.

In the following, we enumerate a list of the possible cases and analyze their possible outcomes.

Let’s consider a simple, unspecified instance of reasoning by analogy as follows:

$[A \sim_f B, g(B) = \beta, f \rhd_\lambda g] \Rightarrow_\lambda g(A) = \beta$

A very simple example will serve as main guideline:

- *Tom is like Ana, performing the same job. The job determines the vote. Ana voted for Claire in the last municipal election. Hence, Tom voted for Claire in the last municipal election.*

What kind of relevant new evidence could challenge this reasoning that we will assume as a basis in everything that follows.

8.1 First-level beliefs

Initially, as previously emphasized, this reasoning is only useful if we don’t get in the background knowledge some first-level belief that makes it redundant, such as:

$\forall X, f(X) = \alpha \Rightarrow_\mu g(X) = \beta$.

In this case, the first level evidence, relevant for the target object, always supersedes the analogical reasoning, which relies on a meta-level assumption. If one knows that f(A) = α, then one knows that g(A)= β.

- *Tom is like Ana, performing the same job. Tom is a lawyer. Lawyers generally vote Republican. Claire is the only Republican candidate. Hence there is no need to know who Ana did vote for to know that Tom voted for Claire.*

But first-level belief may also contradict an analogical reasoning if we believe on the contrary that: $\forall X, h(X) = \alpha' \Rightarrow_{\mu} g(X) = \beta' \neq \beta$ and that h (A) = α'.

In this case, first level evidence indicated that analogical reasoning is defeated for this situation.

- *Tom is Lucie's brother. Lucie was candidate to the election. Generally, people favor their family in an election. Hence Tom did probably not vote pour Claire, even though he has the same job than Ana.*

8.2 Other analogies

New analogies leading to a dynamic that may either contradict or reinforce the conclusions of the initial reasoning are of four types, which are intended to be exhaustive:

1. "Competing analogies" express the fact that C is also analogous to B from another point of view: $C \sim_h B$.
2. "Rival analogies" express the fact that A is also analogous to another object C from another point of view: $A \sim_h C$.
3. "Additional analogies" instead propose a new favorable analogy between the same two objects: $A \sim_h B$.
4. "Disanalogies" show the difference between the two objects from another point of view: NOT $(A \sim_h B)$.

The consequences of these different types of additional analogies are variable and depend on the types of meta-level rules one associate with them. There are mere applications of the general patterns of inference defined in §6.

8.21 Competing analogies

A competing analogy has no effect on reasoning by analogy. The fact that $C \sim_h B$ does not invalidate the first reasoning, even if $h \rhd_{\mu} g$. In any circumstance, it will not change the fact that g(B) = β, and thus can only lead to the non-obtrusive additional conclusion that g(C) = β as well. As an example:

- *Challenging the fact that Bruges is the Venice of the North (because it has many canals) by the fact that the Venice of the North is rather Antwerp (because it is a former economic capital) does not lead to questioning the conclusions of the reasoning one draws from the Bruges / Venice analogy if one deduces them correctly from the point of view taken as reference (its many canals): analogies put in absolute form have no epistemic value but only rhetorical value (arrogating to oneself the title of Venice of the North).*

8.22 Rival analogies

A rival analogy $A \sim_h C$ may instead directly lead to revising the conclusion of the first reasoning. Indeed:

$$[A \sim_h C, g(C) = \gamma \neq \beta, h \triangleright_\mu g] \Rightarrow_\mu g(A) = \gamma \neq \beta$$

In a reasoning by analogy, it may indeed be emphasized that the analogy on which it rests is not the most "relevant" one for driving the conclusion. This intuition may be captured by the respective strength of the determination rules involved. This respective strength will be defined as follows:

If $f \triangleright_\lambda g$, then $\forall\alpha, \exists\beta_\alpha, \forall X, [f(X) = \alpha] \Rightarrow_\lambda g(X) = \beta_\alpha$

If $h \triangleright_\mu g$, then $\forall\alpha', \exists\beta'_{\alpha'}, \forall X, [h(X) = \alpha'] \Rightarrow_\mu g(X) = \beta'_{\alpha'}$

We can then define an order relation between the determination rules written $f >_g h$ which expresses that "f determines g more strongly than h" in the following way:

$f >_g h$ iff $f \triangleright_\lambda g$, $h \triangleright_\mu g$ et $\forall\alpha, \forall\alpha', \forall X, [f(X) = \alpha, h(X) = \alpha'] \Rightarrow_\lambda g(X) = \beta_\alpha$

Everything happens as if the determination by h does not matter.

It is reasonable to accept that when $f \triangleright_\lambda g$, $h \triangleright_\mu g$ then $f >_g h$ iff $\lambda > \mu$ because:

- If $f \blacktriangleright g$ ($\lambda = 1$) or $h \blacktriangleright g$, nothing can change the outcome of the deductive determination rule: deductive reasoning is not defeasible by a non monotonic or a probabilistic rule, and it is not logically possible when $f(A) = \alpha$, $h((A) = \alpha'$, and $\beta_\alpha \neq \beta'_{\alpha'}$ that $f \blacktriangleright g$ and $h \blacktriangleright g$.
- If $f \triangleright g$ and $h \triangleright g$ ($\lambda = \mu = 1\text{-}\varepsilon$), then nothing in the data can serve to decide between the two non-monotonic rules. We ought to wait for new evidence.
- If $f \triangleright g$ and $h \triangleright_\mu g$, with $\mu < 1$ (or the reverse), then the non-monotonic rule prevails on the purely probabilistic one.
- If $f \triangleright_\lambda g$ and $h \triangleright_\mu g$, with $\lambda < 1$ and $\mu < 1$, then the more reasonable is to follow the rule which have the higher probability, which is the one on which one is willing to bet the most.

    - *Tom is also like Ted, of the same age. Ted voted for Joan in the municipal election. We then challenge the conclusion that Tom voted for Claire because he does the same job as Ana. The outcome of this debate depends on whether the determination of vote by age overrides its determination by the job. If it does, Tom's job determination will be erased in the final reasoning.*

But rival analogies could also lead to reinforcement of the initial reasoning in the situation where the comparison with another source object points to the same result for the conclusion:

$$[A \sim_h C, g(C) = \beta, h \triangleright_\mu g] \Rightarrow_\mu g(A) = \beta$$

This situation of "favorable rival analogies" is in fact equivalent to an additional analogy with the same source object, which is described just below, hence it will be more completely analysed in the next section.

8.23 Additional analogies

Additional analogies are frequently cited in the literature as new arguments supporting an analogical reasoning. Bartha (2013, p.19) identifies eight "commonsense guidelines" for evaluating analogical arguments found in logical and philosophical literature (including texts of Aristotle, Mill, Keynes, Robinson, Stebbing and Copi). Two of these guidelines pertain to additional analogies:

(CS1) The more similarities (between the two domains), the stronger the analogy.

(CS8) Multiple analogies supporting the same conclusion make the argument stronger.

Suppose that to the initial analogical reasoning:

$[A \sim_f B, g(B) = \beta, f \rhd_\lambda g] \Rightarrow_\lambda g(A) = \beta$

We add a new one:

$[A \sim_h B, g(B) = \beta, h \rhd_\mu g] \Rightarrow_\mu g(A) = \beta$, with $0 \leq \mu \leq 1$.

As noticed above, the conclusion is the same if B is replaced by another object C, with $g(C) = \beta$.

The final value predicted by the analogical reasoning remains unchanged. What may change in this scenario is merely the nature and degree of belief in the conclusion $g(A) = \beta$. These new facts can enhance the strength of the first analogical reasoning. But this is not provable to be general. Once again, consider the various types of possible determination rules:

- If $f \blacktriangleright g$, no strengthening of this determination rule, and thus of the conclusion based on it, is possible. Additional analogies are in this case useless.
- If $f \rhd g$, a strict additional analogy $h \blacktriangleright g$ overrides the first one, but the reinforcement of the conclusion of the reasoning must not be interpreted as the effect of their combination. A non-monotonic alternative analogy $f \rhd g$ leads to no reinforcement because there is no degree of belief associated with a non-monotonic inference.
- If $f \rhd_\lambda g$ ($\lambda < 1$), and $h \rhd g$ or $h \blacktriangleright g$, then the additional analogy allows for a determination rule that dominates the former and replaces it. Again, strengthening the conclusion of the reasoning is possible but growing the number of considered analogies is not a factor which increases the strength of the first one.

In all the previous scenarios, increasing the number of analogies is not what leads to a strengthening of the conclusion. It only allows in some cases the replacement of a weaker analogy reasoning with a stronger one.

The only situation that may correspond to the intuition of strengthening the reasoning by increasing the number of additional analogies is the one in which $f \rhd_\lambda g$ and $h \rhd_\mu g$, with with $0 < \lambda < 1$ and $0 < \mu < 1$. In this case, the data are as follows:

$f \rhd_\lambda g$, i.e. $\forall X, \forall Y, Pr((g(X) = g(Y)) / (f(X) = f(Y))) = \lambda$

$h \rhd_\mu g$, i.e. $\forall X, \forall Y, Pr((g(X) = g(Y)) / (h(X) = h(Y))) = \mu$

The task here is to compute the value θ of Pr((g(X) = g(Y)) / (f(X) = (f(Y)) & (h(X) = h(Y))). The situations of interest here are those where λ > Pr((g(X) = g(Y)) and μ > Pr((g(X) = g(Y)), meaning that there is a positive relative confirmation, or support, of the conclusion associated with each analogy. The common intuition behind the additional analogy condition is that in these situations θ > Pr((g(X) = g(Y)), and even that $\theta > \lambda$ and $\theta > \mu$. But this is not necessarily the case as the following counter example shows.

- *Let's consider this situation with 6 people:*

Name	Vote	Job	Religion
Bob	Democrat	Physician	Jewish
Ana	Democrat	Physician	Catholic
Lea	Republican	Dentist	Muslim
Ted	Republican	Lawyer	Muslim
Eva	Liberal	Philosopher	Buddhist
Rick	Socialist	Philosopher	Buddhist

Then it is easily checked that:
Pr(same vote / same job) = 0,5 > Pr (same vote) = 0,13
Pr(same vote / same religion) = 0,5 = Pr(same vote) = 0,13
But Pr(same vote / same job and some religion) = 0
The additional analogy "same religion" leads to cancel the support of "same job" to "same vote", even though itself alone supports "same vote".

It is also possible that an additional analogy lessens or nullifies the initial conclusion when the additional analogy is associated with a counter-determination rule: h $\lhd_\mu$ g, where similarity creates differences. In this case one gets:

$A \sim_h B, g(B) = \alpha, h \lhd_\mu g \Rightarrow_\mu g(A) \neq \alpha$

Which contradicts the initial reasoning.

Like the situation with rival analogies, we must compare the respective strengths of the inferences associated with the determination rule and to the counter-determination rule. This can be expressed as follows:

$f >_g h$ iff $f \rhd_\lambda g$, $h \lhd_\mu g$ and $\forall X, \forall Y, \forall \alpha, f(X) = \alpha, h(X) = h(Y) \Rightarrow_\lambda g(X) = \beta_\alpha$

$h >_g f$ iff $f \rhd_\lambda g$, $h \lhd_\mu g$ and $\forall X, \forall Y, \forall \alpha, f(X) = \alpha, h(X) = h(Y) \Rightarrow_\mu g(X) \neq \beta_\alpha$

It is easy to check again that:

When $f \rhd_\lambda g$, $h \lhd_\mu$, $f >_g h$ iff $\lambda > \mu$.

Indeed:
- It is not possible to get simultaneously:
$A \sim_f B$, $A \sim_h B$

f ▶ g, i.e., $\forall X \forall Y$, $f(X) = f(Y) \Rightarrow g(X) = g(Y)$
h ◀ g, i.e., $\forall X \forall Y$, $h(X) = h(Y) \Rightarrow g(X) \neq g(Y)$
Indeed, let us suppose that $f(B) = \alpha$ and $g(B) = \beta_\alpha$:
$A \sim_f B$ et f ▶ g $\Rightarrow g(A) = \beta_\alpha$
$A \sim_h B$ et h ◀ g $\Rightarrow g(A) \neq \beta_\alpha$
This leads to a contradiction.

- A strict determination rule such as f ▶ g would dominate any non-monotonic or purely probabilistic counter-determination rule, and vice versa if we get a strict counter-determination rule h ◀ g.

- When the two rules are non monotonic, i.e., f $\triangleright$ g, h $\triangleleft$ g, it is not possible to conclude, and one must wait for new evidence.

- If one rule is non monotonic and the other purely probabilistic, for instance f $\triangleright$ g and h $\triangleleft_\mu$ g, with $\mu < 1$, the non-monotonic rule dominates the probabilistic one.

- If the two rules are purely probabilistic, i.e., f $\triangleright_\lambda$ g and h $\triangleleft_\mu$ g, with $\lambda < 1$ and $\mu < 1$. Then it is reasonable to choose between the opposite conclusions $g(A) = \beta_\alpha$ and $g(A) \neq \beta_\alpha$ by looking at their respective probabilities, because one has to bet on one of the alternative conclusions.

- *Let's alter the initial analogy by assuming that Ana voted for herself. If Tom, like Ana, was a candidate for mayor in the last municipal election, this analogy is associated with a partial counter-determination rule: they did not vote for the same person, but each for themselves. The outcome of this additional analogy, therefore, does not strengthen the initial analogical reasoning, and in this case, it likely overrides it.*

8.24 Disanalogies

Disanalogies are typically viewed as potential arguments against an analogical reasoning, which can reduce its strength or even invalidate it. In enumerating the commonsense guidelines found in literature for evaluating analogical arguments by Bartha (2013, p.19), the second is:

(CS2). The more differences, the weaker the analogy.

In current exchange between an advocate and a critique of an analogical reasoning, disanalogies will often be the main counterarguments used by the critique.

Assume that NOT $A \sim_h B$. This indicates a difference between A and B from the point of view of the function h.

The association of the disanalogy with another determination rule such as h $\triangleright_\lambda$ g doesn't impact the outcome since, specifically, NOT $A \sim_h B$, so no inference could be derived from this rule.

But one may associate this disanalogy with a separation rule which could defeat the initial reasoning whose conclusion is $g(A)= \beta$.

The issue of the debate again depends on the respective strength of the determination rule and the separation rule. With the same kind of definitions than above, one gets the following data:

Suppose again that:

$[A \sim_f B, g(B) = \beta, f \rhd_\lambda g] \Rightarrow_\lambda g(A) = \beta$

And that:

$[\text{NOT } A \sim_h B, g(B) = \beta, h \,/_\mu\, g] \Rightarrow_\mu g(A) \neq \beta$ (the symbol $/_\mu$ representing a separation rule)

We can define the fact that f dominates h, $f >_g h$, and reciprocally, as follows:

$f \rhd_\lambda g, h /_\mu, g\ f >_g h \Rightarrow_\lambda g(A)= \beta$

$f \rhd_\lambda g, h /_\mu g, h >_g f \Rightarrow_\lambda g(A) \neq \beta$

As in the previous cases, it is reasonable to believe that:

If $f \rhd_\lambda g$ and $h /_\mu g$, then $f >_g h$ iff $\lambda > \mu$,

If $f \rhd_\lambda g$ and $h /_\mu g$, then $h >_g f$ iff $\lambda < \mu$,

which is trivial if λ or $\mu = 1$ and results from the decision to believe in the stronger probability when $\lambda < 1$ and $\mu < 1$.

Hence, not every disanalogie leads to the dismissal of an analogical reasoning. The issue depends on the strength of the separation rule associated with this disanalogy, which is a good way of representing formally what many philosophers refer to as its "relevance".

See those examples:

- *Tom is tall while Ana is small. This disanalogy does not affect the original analogy because size does not separate votes: two people of different sizes may vote identically.*
- *Tom does not live in the same city as Ana. Since there is only one mayor by city, the city of residence separates strictly votes. Hence, the initial analogical reasoning is refuted: Tom could not vote for the same candidate than Ana.*

Finally, a disanalogy associated to a counter-separation rule could on the contrary be in favour of the initial reasoning since in this case as it has been exposed §6:

$[\text{NOT } A \sim_f B, g(B) = \beta, f \backslash_\lambda g] \Rightarrow_\lambda g(A) = \beta$

This would count like an additional analogy, with the output of adding this evidence to the overall argument dependent either on the logical strength of the counter-separation rule (a deductive or non monotonic rule would supersede a purely probabilistic determination rule), or on the computation of the combined probabilities.

9. Conclusion and comparison with some related works

To conclude, we propose to compare the present work with some of the modern philosophical theories of analogy. This comparative analysis does not aim to achieve an exhaustive review, unreachable within this present paper. Our aim is to test the universality of our inference schemas by comparing them with some of the main alternative theories. Contrarily to ours, these theories highlight the diversity inherent to analogy and analogical reasoning. The purpose of this conclusion is to exhibit several examples showing that our analysis can be applied to the diverse situations described by those works.

One main distinction put forward in the literature is between “inductive analogies” and “a priori analogies”, for instance in Trudy Govier (1989), Stephen F. Barker (1989) or Marianne Doury (2009). Inductive analogies pertain to cases where the premise $P(B)$ is known to be true and based on an empirical observation, whereas a priori analogies involve instances where $P(B)$ remains a hypothesis. A typical example of an a priori analogy is Judith Jarvis Thomson’s (1971) violinist example, the violinist being a fictious person about which one does not know any realistic property but only hypothetical ones. According to Govier, inductive analogies are “predictive analogies”: they aim to forecast outcomes about the source A, such as $Q(A)$. On the contrary, since they do not rely on realistic properties, a priori analogies have no predictive roles. They are instead supposed to be employed only for some descriptive purposes.

Because the facts associated with the source object in an a-priori analogy are based on a simple assumption, they are conditionals. Hence, the logical framework of the reasoning can be formalized as follows:

$P(A), P(B) > Q(B)$ therefore $Q(A)$

Where the corner > takes the place of any logical or probabilistic operator for expressing the belief that “if $P(B)$ then $Q(B)$”. Those premises contrast with the one used in predictive analogies: $P(A)$, $P(B)$, $Q(B)$. The conditional nature of the premise associated with B presents indeed a challenge to the effectiveness of a typical analogical reasoning: it becomes impossible to logically detach the conclusion that $Q(A)$.

But one believes that $P(B) > Q(B)$ when B is a fictitious entity only because one holds a general belief such as $\forall X\ P(X) > Q(X)$, which implies it. The example of the fictitious entity B serves only as a psychologically compelling instance, or an illustration, of this general belief. This generic conditional might as well be directly applied to the source A, without necessitating any analogy with a fictitious entity B. Hence, what is called a priori analogies are simply redundant uses of analogies, where the background knowledge is already equipped with first level beliefs which lead to the conclusion and where the analogy is used for a rhetorical or illustrative purpose. This is not another type of analogies.

Another distinction delineated by Brown (1989) and Juthe (2005) is between predictive analogy and proportional analogy. Predictive analogies are described as comparisons between two objects, while proportional analogies denote situations where two objects bear the same relations as two other objects to each other. In §5, we demonstrated that proportional analogies are entirely reducible to simple analogies. However, according to Brown, despite their reducibility to the same logical form, these types of analogies serve different purposes. As expressed by Juthe (1989, p.87), 'the function of predictive analogy is to predict that an object has a certain attribute, whereas the function of proportional analogy is to point out a common principle between two pairs of objects.' Yet, this distinction appears not correlated to the nature of the two kinds of analogy. It hinges on a pragmatic difference in interpreting or using any statement: any statement could be construed or utilized for prediction or to highlight something. There exists no compelling reason, beyond the author's psychology, to exclusively associate conclusions like $Q(A)$ with predictive use and conclusions such as $R(A,B)$ with the highlighting function, or vice versa.

Stephen P. Barker (1989) makes a difference between inductive or non-inductive analogical arguments, akin to what Govier (1989) refers to as a priori analogies. However, Barker doesn't furnish a specific logical structure for these non-inductive arguments, even though he presents them as distinct arguments that are not reducible to deductive or inductive forms. Michael J. Wreen (2007) proposes then a structure he terms 'Form B' for these arguments. Within our framework, 'Form B' can be expressed as follows:

A $\sim_f$ B, g(B) = β
A $\sim_h$ C, g(C) = β
Hence g(A) = β, allegedly with a stronger support than with only one of those premises.

But this is not a different logical form of reasoning by analogy. As exposed previously in §8, it is only a case of favorable rival analogies, with the same logical form than additional analogies. Once the premises are given, it can be handled by the same inference scheme than any other simple analogical reasoning.

Let P refer to the set of properties shared by the source and the target and Q to the property projected from the source to the target. Bartha (2010) considers that there are four types of analogical reasoning, according to their logical forms:

1. Predictive, when $P \to Q$
2. Explicative, when $Q \to P$
3. Functional, when $P \leftrightarrow Q$
4. Correlative, when no direction exists between P and Q

But this classification is useless and based on some confusions:

On one hand, interpreting $P \to Q$ as representing $\forall X, P(X) \to Q(X)$, or even $Pr(Q(X)/P(X)) = \alpha$, poses a significant risk to the understanding of the nature of reasoning by analogy: the risk of redundancy. The only way to escape this risk is to consider that $P \to Q$ has the form of a meta level rule, such as determination rules. This is not explained by Bartha.

On the other hand, all analogical reasoning starts from an analogy to draw a conclusion, entailing the projection of an additional property from the source to the target. The direction of the reasoning itself is always the same. One never relies on the projected property, which is the hypothetical one, to infer the common property, which is the known one. It is of course possible that the determination rule itself result from an abduction, but this does not concern the analogical reasoning itself. Thus, Bartha's analysis confuses the origin and the use of the determination rule. Consequently, the second type of reasoning described ('explanatory') lacks coherence, leading to the conclusion that the third, a compound of the first two, is similarly untenable. Within this context, the added value of the fourth type of reasoning remains unclear.

Several authors, for instance Brown (1995), Juthe (2005), distinguish between "same-domain" and "different-domain" analogies, where 'domain' refers here to the list of compatible predicates associated with the source or with the target object (to be carefully distinguished from the logical definition of domain or point of view as a set of exclusive properties used in the paper). Same-domain analogies involve objects sharing obviously similar properties, like different human individuals in Mill's argument, inferring that others share feelings akin to oneself. Conversely, different-domain analogies involve objects lacking such similar properties. For instance, analogies between the sound and the light domains exemplify different-domain analogies (Bartha, 2010, Chap.1, p.14). In fact, this difference can easily be handled in a unified model as has been shown in §5. By considering the analogy between properties within distinct domains as operating at a higher level where a same-domain property exists over the properties of the first level, it allows for a unified approach, rather than considering them as fundamentally separate types of analogies. The determination rules used for analogical reasoning could then be selected at this higher level. Of course, the strength of this kind of analogical argument may be weaker since it may rely upon more general determination rules, at less empirical levels.

Finally, the philosophical literature contains many different conditions for accepting an analogical reasoning.

For instance, Bartha (2010) quotes what he calls eight commonsense guidelines of analogical reasoning (Part 1, p.19) identified in the literature (numbered CS1 to CS8). We already analyzed CS1 and CS8 in section 8.23 and proved that they can be handled by our theory and are neither necessary neither always correct, despite their intuitive features. Idem for CS2 in section 8.24. It can easily be shown that the other ones are vague principles that can be handled within our model once we give them more precise definitions.

For instance, CS7 states that: "The relevance of the similarities and differences to the conclusion must be taken into account". This highlights the critical need for evaluating our degree of belief in meta level rules, formalizing the intuitive notion regarding the relevance of similarities and differences to drawing a conclusion in analogical reasoning.

Bartha suggests replacing these vague commonsense guidelines by a very complex "Articulation model" built upon three sets of conditions which can be briefly summarized by six conditions:

1. There is a Prior Association (PA) characterized by a "relation" between a list of "factors" - let's say properties of the source object B and the target object A - and the conclusion Q(A).
2. There are some "relevant" factors which have a positive effect on the conclusion.
3. There are no "critical" factors which are associated with disanalogies.
4. The strength of the Prior Association.
5. The scope of the "positive analogy", for instance more critical factors in the positive analogy strengthens the conclusion.
6. The multiplicity of positive analogies strengthens the conclusion, negative analogies may weaken it.

But Bartha's model remains only qualitative and relies again on many vague notions. it lacks a unified formalisation showing how the conditions are to be combined and what is the logical or quantitative connection between the conditions of the model and the conclusion.

Some of the conditions Bartha presents are straightforward outcomes of our analysis. For instance, PA is not formally specified, but we may take it as a simple consequence of our general inference scheme. Conditions 2 and 4 are also natural consequences of the same scheme: the robustness of the conclusion directly corresponds to the strength of the determination rules between the domains, precisely measuring what's termed as positive "relevance" of the influencing factors. Condition 5 appears challenging to differentiate from condition 6 without additional formalization. Moreover, conditions 3, 5, and 6 might be true in specific scenarios, but as demonstrated in our section §8, they are not universally mandatory. Their necessity depends on the meta-level rules employed in the reasoning process, as explained in the present paper. Finally, Bartha's articulation model also misses the essential condition of non-redundancy.

This last key point has been put forward by Davies & Russel (1987), and developed in Walliser, Zwirn & Zwirn (2021). The common point to these works is to avoid explicitly redundancy by using a meta-level background hypothesis. Davies & Russel (1987, p.2) proposed to express determination rules as follows:

$\forall X\ (P(X) \rightarrow Q(X))]$ or $[\forall X\ (P(X) \rightarrow -Q(X))$

But this expression misses the explicit introduction of domains, and the necessity to quantify over the properties of two domains. Moreover, it sticks only to a purely deductive form of determination rules.

Walliser, Zwirn & Zwirn (2021) introduced the concept of domains within the determination rule and the fact that it may be probabilistic. The present paper changes the way to express the probabilistic situation and added the context of non-monotonic determination rules. More importantly, it delves into the application of a general inference scheme of analogical reasoning for analogical debates and broadens the concept of determination rules to meta-level rules that yield either positive or negative effects on conclusion. Rather than viewing analogical debates as distinct cases, this paper emphasizes analogical reasoning as a specialized instance of a broader reasoning process involving similarities and differences—a novel approach. This wider-ranging reasoning encompasses four typical inference schemes but does not constitute a new logic or introduce a distinct fourth mode of reasoning.

Beardsley (1989) posits that any argument from analogy is inherently a fallacy, unless it is regarded as reducible to deductive arguments featuring a missing premise, a hidden generalization. Evelyn M. Barker (1989) and Govier (1989) critique this reduction, citing concerns about redundancy issues it brings forth. This is not a problem for our analysis since one prevents redundancy by using hypotheses of a higher level. In fact, drawing a distinction between deductive and inductive analogies becomes inconsequential: all analogical reasoning formally equates to a singular case induction, as detailed in Walliser, Zwirn & Zwirn (2021). This induction could theoretically be reframed as deductive when employing deductive determination rules within the background knowledge. However, in empirical cases, it generally aligns more with a non-deductive framework.

Other kinds of logic could be incorporated into those inference schemes, for example modal or counterfactual logics, and could lead to some other interesting applications or use cases. It may also be interesting to apply this analysis to some of the classical problems of induction.

Of course, the thesis of universality of our inference schemes remains open to challenge and counter examples and could be deepened by comparisons with more other works.

Ethical Statement

The authors certify that:

- The information contained in this article present the results of their research as well as an objective discussion of these results and their importance. No known fraudulent and consciously inaccurate information is presented.
- This article is a completely original study, and the identification of research done by others is always given.
- They do not submit at the same time an article representing the same results to another journal or book.
- The authors have both made a significant contribution to the present article and they both approve the final version of the text and they have agreed to its submission for publication.
- There are no conflicts of interest that may affect the proposed publication.
- If the authors discover an important error or an inaccuracy in the present publication, they will quickly inform the editor and consider, in agreement with the person in charge, the withdrawal of the article or the publication of the information about the error.

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