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Are the Quantum Rules Exact? The Case of the Imperfect Measurements[1]

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Should we doubt the exactness of the predictive quantum rules of calculation? Although this question is sometimes raised in connection with the one on how to physically understand quantum mechanics, these two questions should not be mixed up. It is recalled here that even the first one is stil an object of controversy, and it is shown (a) that in one specific case the arguments put forward in support of such doubts are hardly cogent but (b) that, nevertheless, at least in one specific other context, the question is worth attention. This is the problem of repeated imperfect measurements. Relative to it, a theoretical possibility is shown of discriminating between the thesis that the quantum rules are exact and a powerful theory of which it is proved that it cannot be reconciled with the assumption.

1. INTRODUCTION

Any discussion bearing on the basic ideas of quantum physics should first of all carefully distinguish between the axioms of this theory and its conceivable interpretations. It should admit that the former are but predictive "rules of computation" and that this is the very reason why they constitute the hard core of the theory in question: for the observation that these rules are well verified in a great number of experiments is just that of a plain fact, which no dispute over "the significance of it all" can falsify.

Once the necessity of making this distinction is acknowledged, the problems bearing on the foundations nicely separate—in a first approximation at least—in two sets. One of these is composed of the problems in which the rules in question are assumed to be both exact and

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universal. Such problems are of the type "How can we interpret these rules in terms of an independently existing reality (of a 'hardware located out there' as Professor Wheeler puts it)?" or even, more radically, "Do we *have* to interpret them in such a way?." As is well known, Wheeler himself has taught us in a masterly way to look boldly at several perplexing possibilities, including even that of a negative answer to this last question. Since the vistas he opened thereby are such as to influence, in the long run, the whole of our thinking on the measurement problem, it is a great pleasure to dedicate to him an investigation such as this one.

The problems in the other set are those in which it is the universality of the quantum rules themselves that is questioned. Of course the possibilities along these lines are limited. Nevertheless they do exist due to the obvious fact that the actual experiments confirming the rules in question have but a finite accuracy and do not cover the infinite domain of what could conceivably be tested (and, of course, solving the problems in the first set would be less difficult if the idea of the universality of the quantum rules was given up). As implied by the title, the problems investigated in this article belong to the second set.

At this stage it is, however, necessary to be somewhat more explicit on one point. It concerns the special status of the "last" quantum mechanical axiom, namely the one concerning such things as the collapse of the state vector upon measurement ("projection postulate"). In fact, although this axiom is often useful in computing experimental predictions, for such a purpose it can always, at least in principle, be dispensed with if the Schrödinger equation is taken as universal. For this and for other reasons the problems concerning it can to a large extent be postponed until the question of the interpretation of the theory is considered, that is, these problems actually belong to the first set. Let us therefore take the last axiom out of the set of what we call "the rules of quantum mechanics." Let us also, by the way, restrict our considerations to the case of nonrelativistic quantum physics. Then, the hypothesis that the quantum rules are exact reduces to the assumption that (a) the Schrödinger time-dependent equation is an exact one, and (b) the probabilities of the various possible outcomes of a measurement are given by the usual probability rule.

For this second axiom to make sense, we have of course to admit that a classically describable "ultimate ideal measuring setup" can be thought of in connection with any of the physical phenomena we are interested in. It is with respect to the measurement results given by this setup, conceived of as placed at the end of the whole process and assumed to operate without failures of any kind, that the probability axiom is supposed to be valid.

Out of the many types of proposals implying or suggesting that the above-defined quantum rules are *not* exact, we here select two for discussion. One of them is the class of the proposals put forward by physicists hoping to thereby save Einsteinian local realism—or more generally "separability"—in spite of the contrary evidence provided by the Bell theorem and by the corresponding distant correlation experiments. We comment on it qualitatively in Sect. 2. The other one is an interesting idea essentially aimed at describing repeated imperfect experiments. In Sect. 3 (where our acceptation of the word "imperfect" is also given) some consequences of this idea are compared with those of a method strictly based on the quantum rules. The results are discussed in Sect. 4.

2. LOCAL REALISM VERSUS QUANTUM RULES

As noted above, many attempts of varied kinds were made at saving Einsteinian local realism in spite of recent evidence. Most of such attempts are based on the fact that the experimental tests of the Bell inequalities have up to now been made with detectors that are poor approximations to ideal ones (in particular, the number of undetected pairs far exceeds the number of detected ones). A consequence of this is that the experimental results actually disprove local realism only if an additional hypothesis is made, relative to the way the instruments work. Although some such hypotheses are very natural, still the upholders of local realism stress the fact that they all are but assumptions. And they go on by pointing out that—as one of them puts it—"most scientists, given a choice between abandoning [local] physical reality and abandoning such an *ad hoc* hypothesis ··· will unhesitatingly choose the latter course."

A priori such statements look sound. And—especially when they appear in print as natural conclusions to debates on these matters, as was recently the case[1]—they may well give an unprejudiced reader the impression that in truth the choice is between relinquishing some technical hypotheses and abandoning a very general principle which up to now has worked well. There would of course then be no doubt as to where the preference of a physicist would go. Actually, however, the dilemna is quite of a different kind: for keeping Einsteinian local realism does not only imply giving up a technical hypothesis about imperfect instruments. It also implies giving up the hypothesis that the rules of quantum mechanics are exact in the sense explicated in the introduction. For if they *are* exact in this sense then no experiment is needed. Bell's theorem alone disproves Einsteinian local realism. The choice therefore is *not* between a technical assumption on the one hand and a general principle on the other hand. Although the experiments made in connection with the Bell inequalities were most useful, their very existence has had the inconvenience of obscuring to some extent the point just made. The real choice is a sharp one and it is between two general basic ideas: the one of local realism (or separability, which amounts to the same within some here irrelevant small differences[2]) on the one hand and the one that the quantum rules are exact on the other hand. But then it must be admitted that the experimental evidence in favor of the latter view has been steadily growing during the seventies and is now really impressive.[3] In fact, if local realism were to be maintained, the presently available data could apparently be accounted for only by means of ingenious theoretical models (see, e.g., Ref. 4) which

(a) are purely *ad hoc* in several respects (they rely on special assumptions about the ways some particles escape detection and, above all, they are by no means derivable from a general physical theory) and

(b) are built up so as to lead to predictions coinciding more or less with the quantum ones as regards photon-pair correlations. This coincidence, however, could only be made approximate, the difference being fortunately not appreciably larger than the present experimental accuracy limit.

Few are anyway the scientists who are so fond of local realism that they would heartily give up, for that reason, the idea that the quantum rules are exact; but presumably fewer still will be those who will take this step when it is fully realized that we are merely offered *such* models as alternatives to the rules in question.

3. IMPERFECT MEASUREMENTS, COMPARING THE MILANO AND THE CONVENTIONAL APPROACHES

This section and the next one are concerned with the problem of how to predict the results of such "imperfect" measurements (definition below) are as likely to occur when an observable is repeatedly measured at small time intervals on a system. This problem happens to be related to the question whether or not the quantum rules are exact. This is due to the fact that one of the most powerful theoretical methods for dealing with such measurements, namely the Milano theory,[5] is based on a formalism in which the state reduction procedure occurring in a measurement is treated in an unconventional way. Now, as mentioned above, this state reduction should not *per se* be considered as being part of the "basic quantum rules" the exactness of which is here under examination. Nevertheless a change in the conventional description of it may conceivably imply also a change in what we agreed to call "the quantum rules" themselves. Specifically, it is the question whether or not the Milano theory compels us to introduce a change of such a basic type that is here to be examined.

From the theorist's point of view, the simplest of all conceivable quantum measurement processes is one with the following property. If before the measurement takes place the quantum system (let us call it S) already is in an eigenstate $|\varphi_i\rangle$ of the quantity B which is to be measured, then, after the process is over, S still is in the eigenstate $|\varphi_i\rangle$ *and* the instrument A contains an unambiguous record of what this state is. If A is considered as a quantum system, the second condition implies that A is itself in a state orthogonal to any other state it would be in, had S initially been in any state $|\varphi_j\rangle$ other than $|\varphi_i\rangle$. By convention let us say that such measurements are "perfect" and that other ones are imperfect. As a rule, measuring instruments are contrived so that we can think of them as being at least "approximately perfect" in this sense. However, in processes in which a measurement is repeated a great many times on a system during a short time interval it is readily seen that the constituting elementary measurements are likely to be imperfect. By considering, for example, a sequence of closely packed Stern–Gerlach devices the reader will easily verify this, but it is the merit of the physicists of the Milano group to have been among the first persons to call attention to the point.

Consequently, these physicists had to change—as we said—the projection postulate somewhat. To investigate whether or not this implies a basic change in what we called "the quantum rules"—and, if yes, to check whether this change is observable—we compare, in what follows, the predictions of the Milano theory, for a specific process, with those that follow from just the conventional "quantum rules." In this specific process the system S is a spin-1/2 precessing around the z axis under the influence of a constant mgnetic field and the x component of which is measured, at one or several time(s), with imperfect instrument(s).

In the conventional approach, if ξ is the "pointer coordinate" of an instrument A, we may assume the time evolution of the $S+A$ system to take place in accordance with the von Neumann scheme:

$$g(\xi)|\varphi_i\rangle \to g_i(\xi)|\varphi_i\rangle \tag{1}$$

where g and g_i are the wave functions of A before and after the interaction respectively (for convenience we assume no degeneracy) and where

$$S_x|\varphi_i\rangle = (-1)^{i+1}|\varphi_i\rangle, \qquad i=1,2 \tag{2}$$

A known example of an $S+A$ system obeying this condition is the one described by the Hamiltonian

$$H = H_0 + H' \tag{3}$$

where H_0 incorporates the interaction of S with the constant magnetic field and where

$$H' = \beta(t)\, S_x(-i\hbar)\, \partial/\partial\xi \tag{4}$$

$\beta(t)$ being equal to zero except during a short "measurement time interval" (t', t'') during which it is so large that H_0 can be neglected. Setting

$$\alpha(t) = \int_{t'}^{t} \beta(\tau)\, d\tau \tag{5}$$

and

$$a_i = (-1)^{i+1} \int_{t'}^{t''} \beta(t)\, dt \tag{6}$$

it is then a trivial matter to check that during the measurement time interval the ket

$$|\psi_i\rangle = g[\xi - (-1)^{i+1}\, \alpha(t)]\, |\varphi_i\rangle \tag{7}$$

is, for any function $g(\xi)$, a solution of the Schrödinger equation. Hence the "von Neumann scheme" (1) holds, with

$$g_i(\xi) \equiv g(\xi - a_i) \tag{8}$$

The "imperfect measurement case" is the one in which the mutual orthogonality of the two g_i's cannot be neglected. Equations (1), (7), and (8) show, for example, that if, immediately before they interact, A is in state $g(\xi)$ and S in state

$$\sum_i c_i\, |\varphi_i\rangle, \qquad i = 1, 2 \tag{9}$$

then immediately afterwards the $S + A$ system is in state

$$\sum_i c_i\, g(\xi - a_i)\, |\varphi_i\rangle \tag{9'}$$

From this result, the probability $p_+ (p_-)$ that, after the interaction, an "ultimate observer" would find the pointer coordinate ξ lying in the positive (negative) half axis is easily found to be

$$p_\pm = \langle \psi_0 |\, U^+ \left(\sum_i |\varphi_i\rangle\, f_i^{(\pm)} \langle \varphi_i| \right) U\, |\psi_0\rangle \tag{10}$$

with

$$f_i^{(+)} = \int_0^\infty |g(\xi - a_i)|^2 \, d\xi, \qquad f_i^{(-)} = \int_{-\infty}^0 |g(\xi - a_i)|^2 \, d\xi \tag{11}$$

here $|\psi_0\rangle$ is the initial spin state and U is the appropriate spin evolution operator.

Now, this formula coincides exactly with the one resulting from the Milano theory [use Eq. (1.28) of Ref. 5, replacing $|q\rangle \, dq \, \langle q|$ by $|\varphi_i\rangle\langle\varphi_i|$ and $\int dq$ by $\sum_i$) applied to the same process. And this might suggest that the latter theory is basically a "theory of imperfect measurements," in which the conventional quantum rules (in our sense, see introduction) are still exact. However, such an assumption must be submitted to further tests. For this purpose let us now consider the case in which S is made to interact successively, at times t_1 and t_2 respectively, with two instruments A_1 and A_2 of type A. Let us ask for the probabilities $p_{\alpha\beta}$ ($\alpha, \beta = \pm$) that after both interaction processes are completed the pointer coordinates ξ_1 and ξ_2 are found with signs α and β respectively (i.e., lying anywhere on the thus defined half-axes). If the quantum rules are exact, the calculation can be made, as above, without the help of a "projection postulate" of any kind, since the state vector $|\psi(t)\rangle$ of the composite $S + A_1 + A_2$ system can be determined for any value of t by applying equations such as (9) and (9′), at each interaction time, to each of the two components of $|\psi\rangle$ at that time, and by using the usual time evolution operator in-between. The calculation is straightforward. Applying then the usual probability rule at a time $t > t_2$ (which of course implies summing over the spin directions since the "ultimate observer" is not supposed to directly observe the spin), one finds

$$p_{\alpha\beta} = \langle\psi_0| \, U^+ \left(\sum_{ii'} |\varphi_i\rangle \, K_{ii'}^{(\alpha,\beta)} \langle\varphi_{i'}| \right) U|\psi_0\rangle \tag{12}$$

with

$$K_{ii'}^{(\alpha,\beta)} = \sum_j \langle\varphi_i| \, \hat{U}^+ \, |\varphi_j\rangle \, f_j^{(\beta)} \langle\varphi_j| \, \hat{U} \, |\varphi_{i'}\rangle \, M_{ii'}^{(\alpha)} \tag{13}$$

and

$$M_{ii'}^{(+)} = \int_0^\infty g^*(\xi - a_i) \, g(\xi - a_{i'}) \, d\xi, \qquad M_{ii'}^{(-)} = \int_{-\infty}^0 g^*(\xi - a_i) \, g(\xi - a_{i'}) \, d\xi \tag{14}$$

In these equations U and $\hat{U}$ are the unitary operators describing the precession of S during the time intervals $(0, t_1)$ and (t_1, t_2) respectively and the other symbols have their above defined meanings.

Now, if the same problem is investigated within the realm of the Milano method the result is formally given by Eqs. (12), (13), and (11) again, but now with Eqs. (14′),

$$M_{ii'}^{(+)} = (f_i^{(+)} \cdot f_{i'}^{(+)})^{1/2}, \qquad M_{ii'}^{(-)} = (f_i^{(-)} \cdot f_{i'}^{(-)})^{1/2} \tag{14′}$$

replacing Eqs. (14). This follows from Eq. (2.7) of Ref. 5, that is, from applying the Milano "generalized projection postulate" to the interaction of S with A_1 in the way explicated in the quoted reference [at this point also, formula (1.28) of this reference must, of course, be adapted to the case of an observable with a discrete spectrum by making the changes indicated here below Eq. (11)].

4. DISCUSSION

The $p_{\alpha,\beta}$ given by Eq. (12) are different according to whether Eqs. (14) (conventional approach, quantum rules "exact") or Eqs. (14′) (Milano method) are used. Since these $p_{\alpha\beta}$ are observable quantities, the question of principle is settled. The theory based on Eq. (2.7) of Ref. 5 with $\mathbf{I}_{1,2} = \mathbf{R}_{\pm}$ cannot be reconciled with the assumption that the quantum rules are exact, not even in the restrictive sense mentioned in the introduction. This, after all, is not very surprising since this theory was initially based on Ludwig's proposal of a modification of the quantum formalism aimed at removing the difficulties of the measurement theory. But *a priori* the above-considered conjecture that the only real novelty of the method lay in its way of generalizing the projection postulate was not absurd. We now have a proof that its departure with conventional quantum mechanics is in fact much more radical.

Of course this result raises the question "which one of the two methods is correct?" And this brings forward the further question "can the matter be settled experimentally?" On these two questions we can unfortunately make, up to now, but rather sketchy remarks. One of them is that, for such purposes, instruments isolated enough from their environment so as to be describable quantum mechanically would have to be found. This aim is perhaps not unreachable considering the fact that instruments parts of which at least must be so described seem to be necessary for such purposes as that of detecting gravitational waves, and that theoretical investigations incorporating such notions are presently well under way (see, e.g., Ref. 6). Our other remark is that anyhow the problem may be experimentally difficult for, in most circumstances, the observable differences between the predictions derived from Eqs. (14) and (14′) are very small. One way of maximizing the contrast between the predictions of the two theories might be to consider the difference

$$d = p_{+,+} - p_{+,-}$$

From formulas (10)–(14), (14′) d is found to be (initial state $S_x = +1$)

$$d = \cos 2\theta (f_1^{(+)} - f_2^{(+)})(\cos^2 \alpha M_{1,1}^{(+)} - \sin^2 \alpha M_{2,2}^{(+)}) - \sin 2\theta \sin 2\alpha \, M_{1,2}^{(+)} \quad (15)$$

where 2α and 2θ are the precession angles corresponding to intervals $(0, t_1)$ and (t_1, t_2) respectively. Since $M_{1,1}^{(+)}$ and $M_{2,2}^{(+)}$ are identical in the two theories, it is appropriate to choose $\theta = \pi/4$. Then, with

$$g(x) = \pi^{-1/4} \exp(-x^2/2) \quad (16)$$

one gets (with $a = a_1$, $a_2 = -a_1$ and assuming $a \gg 1$)

$$d \propto \exp(-a^2) \text{ in the conventional theory and}$$

$$d \propto a^{-1/2} \exp(-a^2/2) \text{ when Eq. (14′) is used}$$

With the same notations and still assuming $S_x = +1$ the probability p that the first instrument pointer coordinate is found negative when both the magnetic field and the second intrument are removed is

$$p \propto a^{-1} \exp(-a^2)$$

hence, as a function of a, the ratio $d/p^{1/2}$ tends to a constant limit for $a \to \infty$ in the second approach while it decreases rapidly in the conventional approach [qualitatively this remains true also for other forms of $g(x)$]. Differences such as this one might conceivably serve to experimentally discriminate between the two theories. This, however, is, as we said, just a sketchy indication and it is clear that much more extensive studies would be necessary before it could be concluded that tests along such lines are feasible.

As a final remark we would like to point out that the theory of imperfect measurements is presumably still in its first stages, that it may become quite useful in connection with that of repeated measurements (with one or with several instruments), and that it may help to open new vistas concerning the question as to what "measurement" actually is.

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NOTE ADDED IN PROOF

The difference between the predictions of the two methods is seen to vanish when, in Eqs. (11) and (14), the summation ranges are shrinked to infinitesimal intervals, the $p_{\alpha\beta}$ being then obtained by adding all the contributions from the relevant intervals. The replacement of Eq. (2.7) by Eq. (2.21) in Ref. 5 amounts to the same. The thus restored agreement indicates that in a sense this replacement is equivalent to a come-back to the conventional quantum formalism.

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